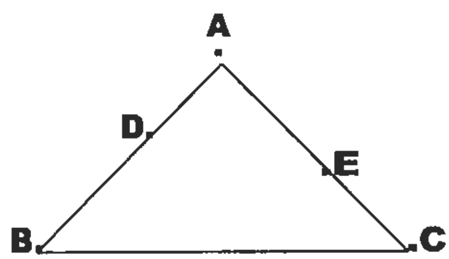
Marshall University Mathematics Competition
Solutions
2004

1. a) Many solutions are possible. The answer below is just an example.
The stack can be reordered in 6 flips by
$241635 \to 614235 \to 532416 \to 142356 \to 412356 \to 321456 \to 123456$

b) Use the first flip to move pancake 6 to the top and then flip the whole stack. Move 5 to the top and reverse the top 5 pancakes. By repeating a similar process for 4, 3, and 2 the stack would be reordered in 10 flips.

2.



ΔABC is isosceles with $\overline{AB} = \overline{AC}$, $m(\angle BAC) = 20°$, $m(\angle BCD) = 60°$, and $m(\angle CBE) = 50°$. What is $m(\angle BED)$?

<u>Solution</u>: $m(\angle ABC) = m(\angle ACB) = \frac{180° - 20°}{2} = 80°$, so $m(\angle ABE) = 30°$ and $m(\angle ACD) = 20°$

$m(\angle BDC) = 180° - 60° - 80° = 40°$ and $m(\angle CEB) = 180° - 50° - 80° = 50°$

Let O be the intersection of $\overline{BE}$ and $\overline{DC}$. $m(\angle DOB) = 180° - 30° - 40° = 110°$

Also, $m(\angle EOC) = m(\angle DOB) = 110°$

Then $m(\angle BOC) = m(\angle DOE) = 180° - 110° = 70°$.

This gives the following system of equations:

$$\begin{cases} m(\angle ADE) + m(\angle AED) = 180° - 20° \\ m(\angle EDC) + m(\angle BED) = 180° - 70° \\ m(\angle ADE) + m(\angle EDC) + 40° = 180° \\ m(\angle AED) + m(\angle BED) + 50° = 180° \end{cases}$$

Unfortunately this system has an infinite number of solutions. We need a different approach.

Using the Law of Sines on ΔBCD we get $\frac{\sin 40°}{\overline{BC}} = \frac{\sin 80°}{\overline{DC}}$.

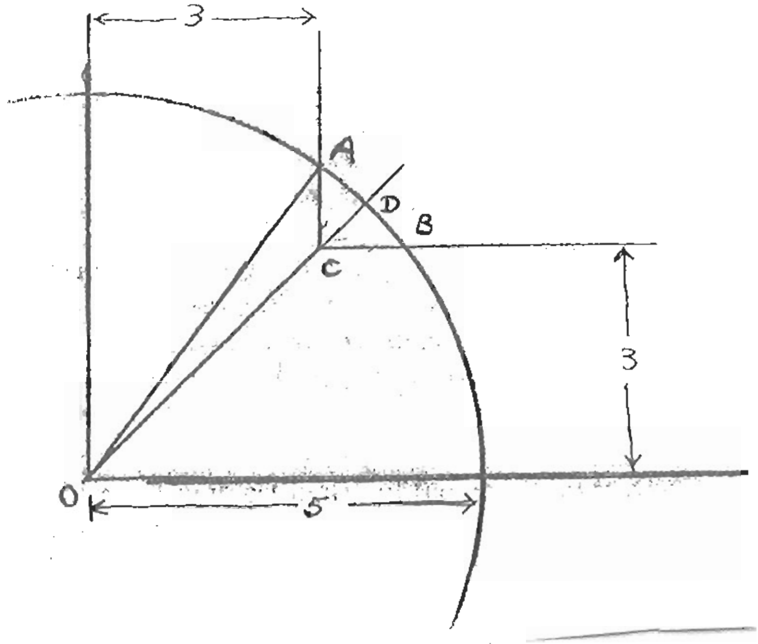
Thus $\overline{DC} = \frac{\overline{BC}\sin 80°}{\sin 40°} = \frac{\overline{BC} \cdot 2\cos 40° \sin 40°}{\sin 40°} = 2 \cdot \overline{BC} \cos 40°$

Let $x = m(\angle BED)$ and $y = m(\angle DEC)$. Then, using the Law of Sines on ΔDEC

gives $\frac{\sin(x+50)}{\overline{DC}} = \frac{\sin y}{\overline{EC}} = \frac{\sin(180-50-20-x)}{\overline{EC}} = \frac{\sin(110-x)}{\overline{BC}} = \frac{\sin(x+50)}{2 \cdot \overline{BC}\cos 40°}$

Multiplying by $\overline{BC}$ gives $\sin(110 - x) = \frac{\sin(x+50)}{1.532}$. Solving with a calculator produces $x = m(\angle BED) = 80°$

3.



On the ceiling consider lines parallel to the walls at a distance of three feet into the room.

Let O represent one corner of the ceiling, and let C represent the point (nearest O) on the ceiling where the parallels to the wall meet each other. Construct a circular arc of radius 5 centered at point O. The circular arc intersects the parallels at points A and B, near C. Extend line OC to meet the circle at point D.

The permissible region is a rectangle with "bites" removed from each corner.

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a. By geometry point A has coordinates (3,4) and B has coordinates (4,3). (If not, simply reverse the roles of A and B.) Point C has coordinates (3,3).

Roughly, the triangle ABC has nearly the same area as convex figure ABDC. ABC is a right triangle, with legs of length 1 . So ABC has area ½. Therefore, convex figure ABDC has area approximately ½. Together the four congruent convex figures have area approximately 2. Therefore the permissible area is roughly $(20 - 6)(15 - 6) - 2 = 124$.

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b. Circular sector AOD has area $\frac{1}{2} r^2\theta$, where r = radius = 5 and θ is the central angle AOD measured in radians. Angle AOD is $\arcsin(4/5) - \pi/4$. Therefore sector AOD has area $\frac{1}{2}\, 5^2\, (\arcsin(4/5) - \pi/4)$

Triangle AOC can be viewed as having AC as its base (length = 1), and then its altitude has length 3 (between C and the point (0,3)). So triangle AOC has area 3/2.

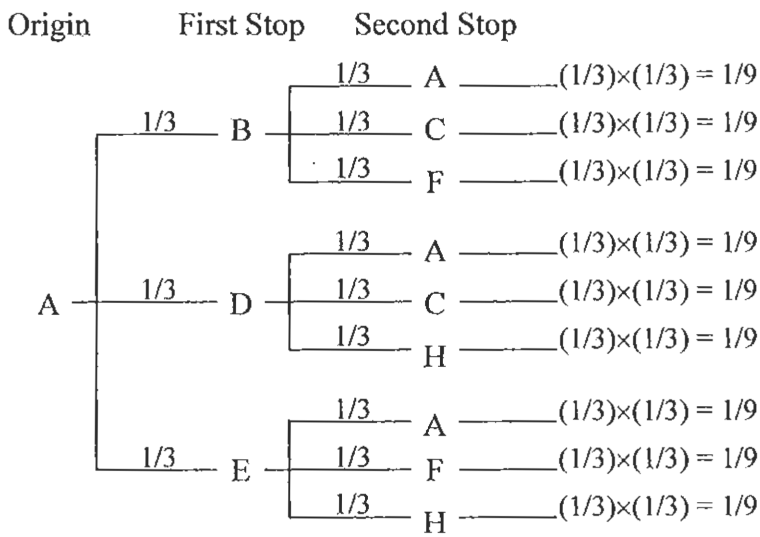
Thus convex region ACD has area $(25/2)\,(\arcsin(4/5) - \pi/4) - (3/2)$.

The four corner bites are 8 copies of convex region ACD. Their total area is $100(\arcsin(4/5) - \pi/4) - 12$

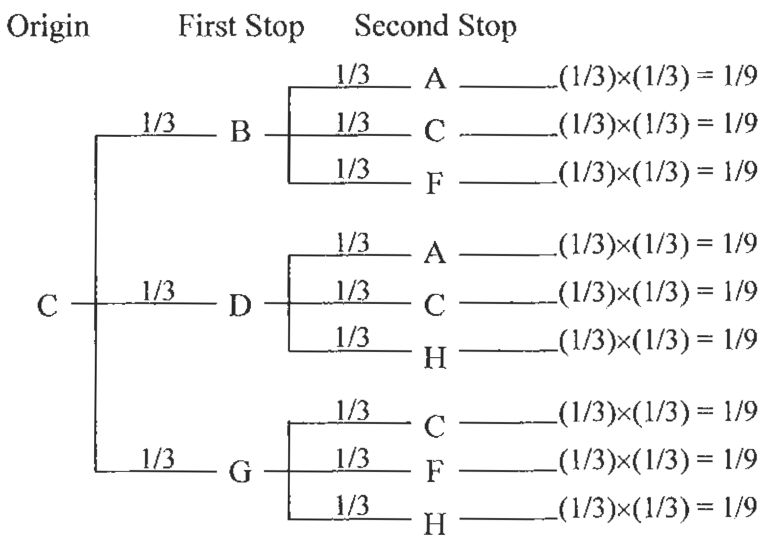
The permissible region has area $(20 - 2\times3)(15 - 2\times3) - (100(\arcsin(4/5) - \pi/4) - 12)$. In decimals this is approximately 123.81029454 .

4. Solutions:

(A) At the end of one leg, she will be in City B, City D, or City E, each with probability 1/3. From any of these cities, she will again have three choices, each with probability 1/3. This gives a total of nine possible routes, each with probability 1/9. These choices are listed in the diagram below. From this diagram, it can be seen that the probability she ends up in City A is 3/9 = **1/3**.



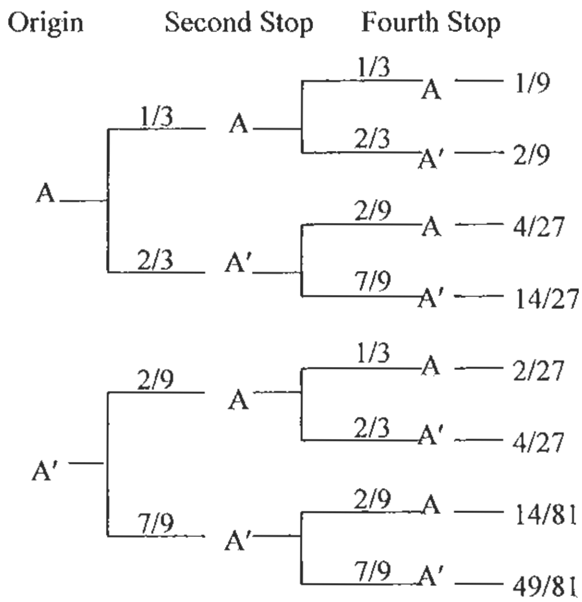
(B) We already know the probabilities for the second stop if she starts in City A. The diagram below gives the probabilities if she starts in City C.



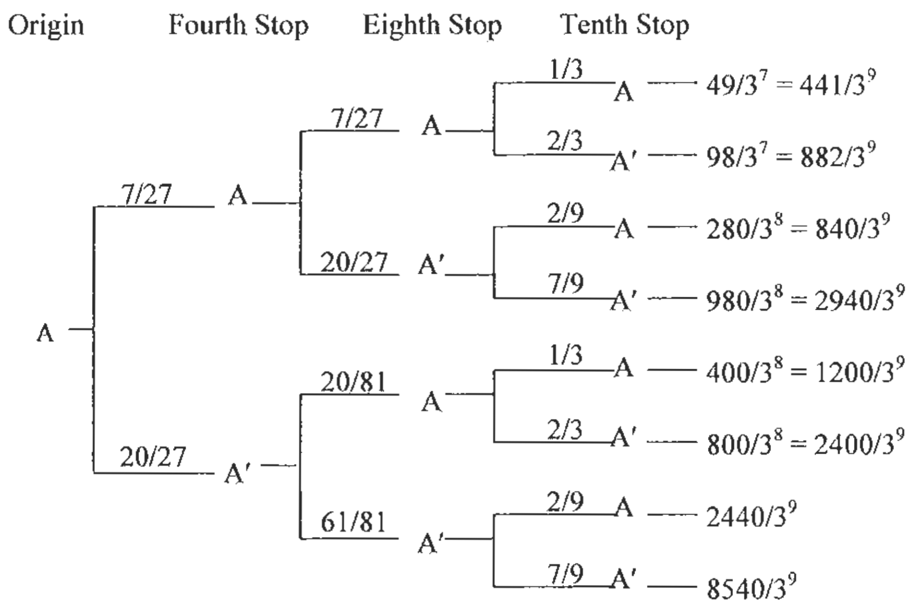
This diagram shows that, from City C, the probability of ending up in city A in two legs is 2/9. Similar diagrams show that the probability is the same from City F and

City H. From this we can conclude that, if she starts in one of A, C, F, or H, she must return to this same set of cities at the end of two legs.

Now where could she end up after four legs? Obviously, we only need to consider the cities A, C, F, and H. In the diagrams below, let A′ represent "not A", which means "C, F, or H". From these diagrams we can see that, starting at A the probability of returning to A in four legs is 7/27; and starting at C, F, or H, the probability of ending up at A after four legs is 20/81.



Now we can combine all this information to obtain probabilities for ten legs. The diagram below gives these probabilities.



From this diagram, we see that the probability of returning to City A is given by

$$\frac{441+840+1200+2440}{3^9} = \frac{4921}{3^9} = \frac{4921}{19683} = \mathbf{0.2500127}.$$

(C) In order to guarantee that she will be able to return home, she must plan a return route after **eight** legs. At this point she will have to be in one of City A, City C, City F, or City H. From any of these cities she can return home in two legs. If she waits until the end of the ninth leg, she will be in one of City B, City D, City E, or City G. If she were to end up in City G, she would not be able to get back to City A in a single leg.

5. Since A, B, and C must be positive to be the lengths of the sides of a triangle, $x > -\frac{1}{2}$ *where* $x \neq 0$. We also know that the sum of two of the sides of a triangle must be greater than the remaining side. That means we can eliminate all values where A + B ≤ C, A + C ≤ B and B + C ≤ A. When the values that make A + B ≤ C or $x^2 + 2x + 1 \leq x + 2$ are eliminated $x > \frac{-1+\sqrt{5}}{6}$. No new values are eliminated when A + C ≤ B or $x^2 + x + 2 \leq 2x + 1$. After eliminating the values that make B + C ≤ A or $3x + 3 \leq x^2$ the only possible range of values for x is $\frac{-1+\sqrt{5}}{2} < x < \frac{3+\sqrt{21}}{2}$.

6. One way to do this problem is to use the identities

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$ and

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ repeatedly.

7. One way to solve the problem is to solve for all the variables in terms of a single variable. In the solution below all the variables are solved for A.

$100D = ADG \rightarrow G = \frac{100}{A}$; $100D = BDF \rightarrow F = \frac{100}{B}$;

$100D = CDE \rightarrow E = \frac{100}{C}$; $50AB = 2FG \rightarrow B = \frac{20}{A}$; $F = 5A$:

$50CF = 2BE \rightarrow C = \frac{4}{A}$; $ADG = 50AB \rightarrow D = 10$; $E = 25A$

To keep A, B, C, D, E, F, and G positive integers, A = 1, 2, or 4.

a) D = 10 b) E = 25, 50, or 100

8. Since $\log_3 cde = 4$, $3^4 = cde$ or $cde = 81$. We also know that c, d, and e being roots of the equation make $10x^3 + 5ax^2 + bx + a = 10(x-c)(x-d)(x-e)$. If the factors on the right were expanded $a = -10cde$. $a = -10(81) = -810$.