2014 Marshall University Mathematics Competition

Solutions

1.

Solution The circular sector has area $A_1 = \dfrac{146 * 95^2 * \pi}{360}$ feet^2.

The portion of the infield below the dash-dotted line has area $A_2 = \dfrac{1}{2} 90^2$ feet^2.

The remaining field consists of equal two triangles each with area $A_3 = \dfrac{1}{2} * 95 * \sqrt{2 * 90^2} * \sin(32^\circ)$ feet^2.

Therefore, the baseball field has area

$A = A_1 + A_2 + 2 * A_3$ feet^2.

2.

Solution (a) Using $a < b$, we have $a < a + a < a + b < b + b$. Dividing by two gives

$$a < \frac{a+b}{2} < b.$$

(b) Using $0 < a < b$, we have $0 < a^2 < ab < b^2$. Taking the positive square root preserves inequalities. Taking the positive square root gives

$$a < \sqrt{ab} < b.$$

(c) Define $\alpha = \dfrac{1}{a}$ and $\beta = \dfrac{1}{b}$. Because $0 < a < b$ we know $0 < \beta < \alpha$. Using part (a), $\beta < \dfrac{\alpha + \beta}{2} < \alpha$.
Therefore we have

$$a < \frac{1}{\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)} < b,$$

meaning $a < h < b$.

3. (a) Answers will vary. The important thing to notice is that 4q + 1 and 4q + 3 will be used for odd integers.

(b) $(4q + 1)^2 = 8(2q^2 + q) + 1$ Since the integers are closed under addition and multiplication, when q is an integer so is $2q^2 + q$

$(4q + 3)\ 2 = 8(2q^2 + 3q + 1) + 1$ Again $2q^2 + 3q + 1$ must be an integer.

4. If a_i represents the number of loaves of bread received by person i,

$a_1 + a_2 + a_3 + a_4 + a_5 = 100$. Since the number of loaves must form an arithmetic sequence $a_2 = a_1 + d$, $a_3 = a_1 + 2d$, $a_4 = a_1 + 3d$, $a_5 = a_1 + 4d$ where d is the common difference. This makes $5a_1 + 10d = 100$. We also know $a_1 + a_2 = \frac{1}{7}(a_3 + a_4 + a_5)$.

So $16 a_1 + 8d + 100$. Solving $5a_1 + 10d = 100$ and $16 a_1 + 8d + 100$ simultaneously makes $a_1 = \frac{5}{3}$ and $d = \frac{55}{6}$. The 5 people would receive $1\frac{2}{3}$, $10\frac{5}{6}$, 20, $29\frac{1}{6}$, $38\frac{1}{3}$ loaves of bread respectively

5. $$\frac{x}{x^2-5x+6} = \frac{A(x-D)+C(x-B)}{x^2-Bx-Dx+BD}$$

This tells us that $B+D=5$ and $BD=6$

If $B=2$, then $D=3$.
If $B=3$, then $D=2$.
Since $B(5-B)=6$

Suppose we use $B=3$ and $D=2$

$$x = A(x-D)+C(x-B) = A(x-2)+C(x-3)$$
$$= (A+C)x - (2A+3C)$$

So $A+C=1$ and $2A+3C=0$

This means $C=-2$ and $A=3$

6. Recommendations may vary.

7. If A and B are inverses their product should be the identity. $x = z = 0$. $y = 4$

8 **ANSWER**

(A) Use quadratic probing to place the following numbers into the table, in the given order: 15, 8, 18, 14. Explain every step of your reasoning.

(Step 1) $15 = 2(6) + 3 \rightarrow$ Place 15 in slot 3.

(Step 2) $8 = 1(6) + 2 \rightarrow$ Place 8 in slot 2.

(Step 3) $18 = 3(6) + 0 \rightarrow$ Place 18 in slot 0.

(Step 4) $14 = 3(6) + 2 \rightarrow$ 2 is occupied.

Try $2 + 1^2 = 3 \rightarrow$ 3 is occupied.

Try $2 + 2^2 = 6 = 1(6) + 0 \rightarrow$ 0 is occupied.

Try $2 + 3^2 = 11 = 1(6) + 5 \rightarrow$ Place 14 in slot 5.

(B) Prove that it will be impossible to use quadratic probing to place 26 into an unoccupied slot.

(Step 5) $26 = 4(6) + 2 \rightarrow$ 2 is occupied.

Try $2 + 1^2 = 3 \rightarrow$ 3 is occupied.

Try $2 + 2^2 = 6 = 1(6) + 0 \rightarrow$ 0 is occupied.

Try $2 + 3^2 = 11 = 1(6) + 5 \rightarrow$ 5 is occupied.

Try $2 + 4^2 = 18 = 3(6) + 0 \rightarrow$ 0 is occupied.

Try $2 + 5^2 = 27 = 4(6) + 3 \rightarrow$ 3 is occupied.

Try $2 + 6^2 = 38 = 6(6) + 2 \rightarrow$ 2 is occupied.

Note that the sequence of attempts is: 2, 3, 0, 5, 0, 3, 2. So we're back where we started.

Prove that the sequence of attempts will just start over now.

$$b_0 + (k+6)^2 = b_0 + k^2 + 12k + 6^2 = 6(2k+6) + (b_0 + k^2) \equiv b_0 + k^2 \pmod 6.$$

So the attempts cycle over the same sequence of 6 numbers over and over. We will never find an unoccupied slot, because 0, 2, 3, and 5 are all occupied.

(C) Consider the general case of quadratic probing for a table with m slots. Derive a rule for determining the minimum number of slots that must be checked in order to verify that a number cannot be placed into the table. Be as specific as possible.

Using the same reasoning as above, it can be shown that the attempts will cycle over the same sequence of m numbers over and over.

$$b_0 + (k+m)^2 = b_0 + k^2 + 2mk + m^2 = m(2k+m) + (b_0 + k^2) \equiv b_0 + k^2 \pmod m.$$

So we will never need to make more than m+1 attempts. But we can do better than that.

Suppose that $2j$ and j^2 are multiples of m. Then

$$b_0 + (k+j)^2 = b_0 + k^2 + 2jk + j^2 \equiv b_0 + k^2 \pmod m,$$ since $2jk + j^2$ is a multiple of m.

So all we have to do is find the smallest number j for which $2j$ and j^2 are multiples of m. (This can be done through prime factorization.) Then j+1 attempts will work.

But we could also take advantage of the symmetry of the pattern in order to cut the number roughly in half:

$j/2 + 1$ if j is even

$(j+1)/2$ if j is odd

In some special cases, it may be possible to cut the number even more. Can you think of a way?