Problem 1: Marco is tied to a corner of a 100′ × 100′ square building in the middle of a large field.

a. How much area can he graze if his leash is 40′ long?
b. How much area can he graze if his leash is 140′ long?
c. How much area can he graze if his leash is 240′ long?

Answer:

a. The region is three-fourths of a circle of radius 40, which has an area of: ¾(1600π) = 1200π ≈ 3770 square feet.
b. This case, his leash is long enough that he can go around two other corners of the buildings, where the leash effectively becomes 40 feet. The entire region consists of three-fourths of a circle of radius 140, and two-fourths of a circle of radius 40, which has an area of: ¾(19600π) + ½(1600) = 15500π ≈ 48700 square feet.
c. This case, he can go around more than half of the building, which means that Marco's regions of going either direction around the building overlap. The entire region consists of three-fourths of a circle of radius 240, two sectors of radius 140 that stop where the circles intersect, and a concave quadrilateral with sides 100, 100, 140, and 140.

The angle of the sectors can be found by drawing a line, from the center of the building to the intersection of the sectors, and a diagonal across the building, which forms right triangles. Angle α of the sector and adjacent angle β of the right triangle form a 135° angle, or 3π/4 radians.

Also, $\cos\beta = \frac{100\sqrt{2}/2}{140}$ so, $\alpha = \frac{3\pi}{4} - \cos^{-1}\frac{5\sqrt{2}}{9}$

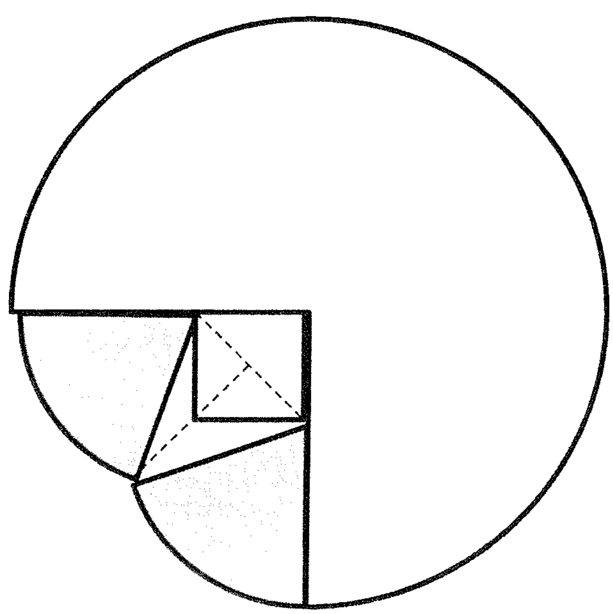
The height of the right triangle is $\sqrt{140^2 - (50\sqrt{2})^2} = 10\sqrt{146}$

and its area is $\frac{1}{2}(50\sqrt{2})(10\sqrt{146}) = 500\sqrt{73}$.

The quadrilateral is the two right triangles with the half of the square building removed.

The area of the entire region is: $\frac{3}{4}(57600\pi) + 2\left(\frac{1}{2}19600\alpha\right) + 2\left(500\sqrt{73}\right) - 5000$

$= 43200\pi + 19600\alpha + 1000\sqrt{73} - 5000 \approx 172000$ square feet.



Problem 2: We can describe the sequence of Fibonacci numbers 0, 1, 1, 2, 3, 5, ... by $f_0 = 0, f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$.

Let the matrix M be $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

a) Compute M^2 and M^3.

b) Show that for all $n > 0$, that $M^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$.

Solution:

a). $M^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $M^2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

b) The statement is true by definition if $n = 1$. If it is true for n-1, then

$M^{n-1} = \begin{pmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{pmatrix}$. So $M^n = \begin{pmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_{n-1} & f_{n-2} + f_{n-1} \\ f_n & f_{n-1} + f_n \end{pmatrix} =$

$\begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$. So, by mathematical induction , the statement is true for all positive n.

Problem 3: Let a, b, c, and d be positive integers so that $\frac{a}{b}$ and $\frac{c}{d}$ are each in lowest terms and $\frac{a}{b} < \frac{c}{d}$. The mediant is defined of the two fractions to be $\frac{a+c}{b+d}$.

a) Find the mediant of 2/3 and 5/7 and show it is in between 2/3 and 5/7.

b.)Show that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Solution: a) The mediant is 7/10. 2/3 = .66666...< 7/10=.7 < 5/7 =.714285.....

b) Because $\frac{a}{b} < \frac{c}{d}$, the difference $\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$ is positive. So $bc - ad > 0$.

Now, $\frac{a+c}{b+d} - \frac{a}{b} = \frac{ab+bc-ab-ad}{(b+d)b} = \frac{bc-ad}{b(b+d)} > 0$, and , $\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc+cd-ad-cd}{(b+d)d} = \frac{bc-ad}{d(b+d)} > 0$ as well.

Problem 4

A Halloween costume kit contains 4 different moustaches, 2 different sets of eyebrows, 3 different noses, 4 different hats, and a set of glasses.

a. How many distinct costumes can be made using at least one of the items?

Solution: There are a couple ways students could think through this.

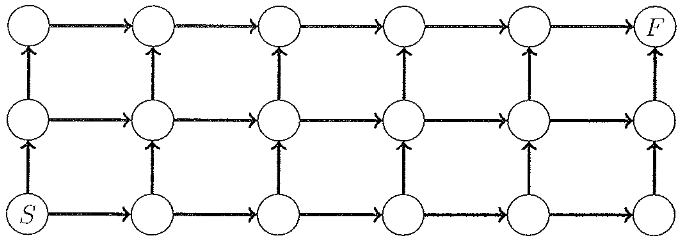
- $5 * 3 * 4 * 5 * 2 - 1 = 599$ obtained because there are 5 options for moustaches (4 moustaches or none), 3 for eyebrows, etc. Since you must wear at least one item, the -1 comes from eliminating the possibility that none of the items were chosen.
- They could list out their options in several ways. Certainly, the number of choices is prohibitive to just do an exhaustive list, but they might separate by how many items, try listing some out and see how to count, etc. We could also make the numbers small enough that the students could simply list everything out.

b. The power went out as you were getting ready and you need to make sure you bring at least two different costume items with you (for example, two sets of eyebrows won't work, but a moustach and eyebrows would). You can't see which items you are choosing and they are all roughly the same size, shape, and weight. How many items do you need to take to *ensure* that you have two different costume items?

Solution: You need to choose at least 5 items. If you have very bad luck, you might choose all the moustaches (or hats), but on the fifth selection, you will get some other costume item.

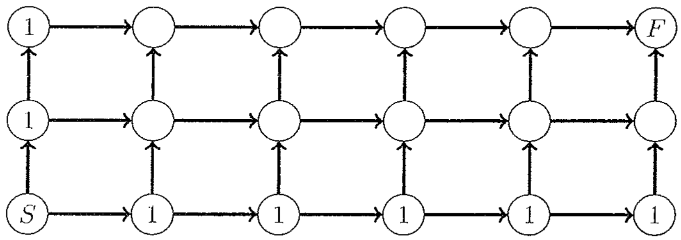
Problem 5

Question (a): Suppose you start at the node labeled S in the graph below. Each arrow indicates a path from one node to another. How many different paths from S to F are possible?



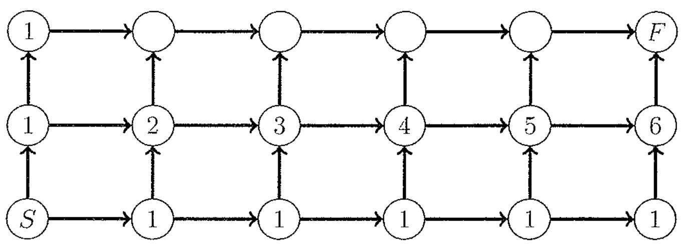
Answer: For convenience, let (i, j) denote the node which is i units right of S and j units up from S.

First, observe that there is only 1 path which can take you from S to any node directly above or to the right of S. We indicate this with a 1 in the node:

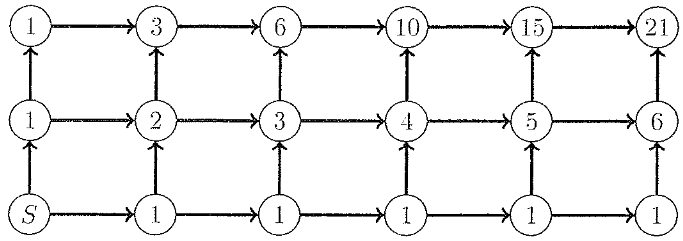


Consider the node at $(1, 1)$. There are two nodes which can travel to $(1, 1)$: those are $(0, 1)$ and $(1, 0)$. There is only 1 way to reach each of these two nodes. Therefore, summing the number of paths through these nodes gives the number of paths through $(1, 1)$, which is $1 + 1 = 2$.

Similarly, there are two nodes which travel to $(2, 1)$: those are $(2, 0)$ and $(1, 1)$. So the number of paths traveling through $(2, 1)$ must be the sum of the number of paths traveling through these two nodes, which is $1 + 2 = 3$. We can continue this to fill out the next row of nodes:



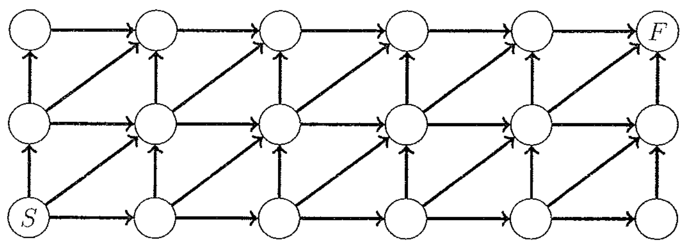
Last, we continue this process for the last row, and the number found in $(5, 3)$ (or F) is the total number of paths from S to F:



So the number of paths from S to F in this graph is 21.

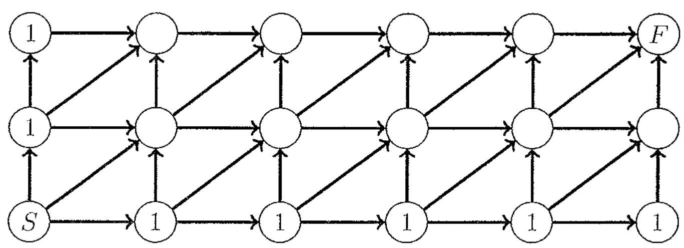
Alternate Answer: To reach F from S, one must travel right 5 steps and up 2 steps, so an example of a path from S to F would be $URRRURR$. In fact, the number of rearrangements of this word is precisely the number of walks from S to F, making a total of $\binom{7}{2} = 21$ paths.

Question (b): Suppose you start at the node labeled S in the graph below. Each arrow indicates a path from one node to another. How many different paths from S to F are possible?



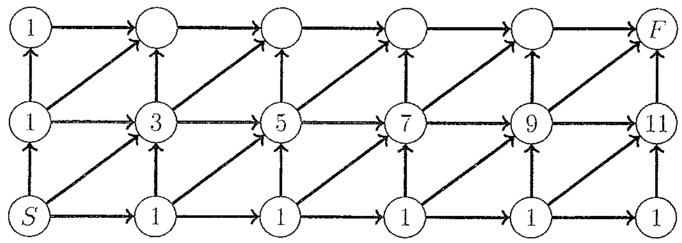
Answer: For convenience, let (i, j) denote the node which is i units right of S and j units up from S.

First, observe that there is only 1 path which can take you from S to any node directly above or to the right of S. We indicate this with a 1 in the node:

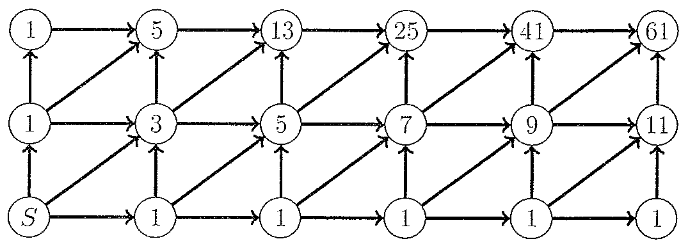


Consider the node at $(1, 1)$. There are three nodes which can travel to $(1, 1)$: those are $(0, 1)$, $(0, 0)$, and $(1, 0)$. There is only 1 way to reach each of these three nodes. Therefore, summing the number of paths through these nodes gives the number of paths through $(1, 1)$, which is $1 + 1 + 1 = 3$.

Similarly, there are three nodes which travel to $(2, 1)$: those are $(2, 0)$, $(1, 0)$, and $(1, 1)$. So the number of paths traveling through $(2, 1)$ must be the sum of the number of paths traveling through these three nodes, which is $1 + 1 + 3 = 5$. We can continue this to fill out the next row of nodes:



Last, we continue this process for the last row, and the number found in $(5, 3)$ (or F) is the total number of paths from S to F:

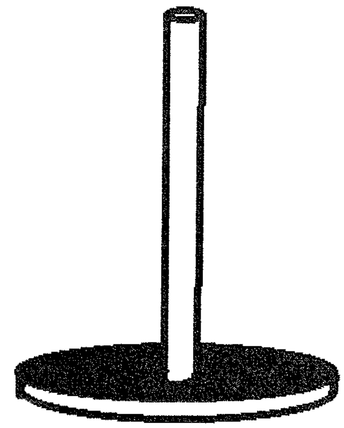


So the number of paths from S to F in this graph is 61.

Because of the northwest direction that can be taken, this cannot be modeled with a binomial coefficient.

Problem 6

An object consists of a flat circular disk attached to a rod made of some other material. The rod is perpendicular to the disk, and is attached to the surface of the disk at the disk's center. The rod and the disk cannot be separated, and the object weighs 8.6 pounds. The disk is 11 inches in diameter and 1 inch thick. The rod is 1.25 inches in diameter and 35 inches long from the surface of the disk out to the rod's tip. The object will balance along the rod at a point 33 inches from the tip.



Our goal is to find the weight of the disk without breaking the object.

a. Choose variable names and state clearly what each one represents.

b. Create one or more equations which will help us to achieve our goal.

c. Solve the equation(s) and clearly state the weight of the disk.

Solution:

a. Let D be the weight of the disk, and let R be the weight of the rod.

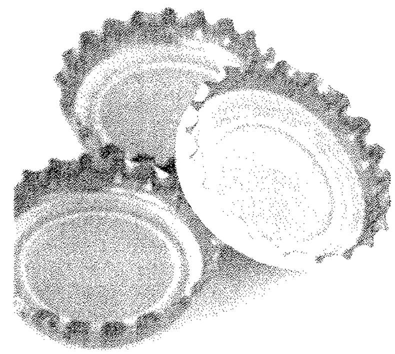
b. Then $\boxed{D + R = 8.6}$ is one equation.

From physics, an object acts as if all its mass is concentrated at its center of mass. The rod's center of mass is 35/2 = 17.5 inches from the tip of the rod. The disk's center of mass is at its geometric center under the rod and ½ inch below the top surface of the disk. So the disk's center of mass is 35.5 inches from the rod's tip. The rod's centroid is 33-17.5 = 15.5 inches above the balance point. The disk' centroid is 35.5 – 33 = 2.5 inches below the balance point. Equating moments around the balance point gives the equation $\boxed{2.5D = 15.5R}$.

All of this gives two equations in two unknowns: $\left\{\begin{matrix} D + R = 8.6 \\ 2.5D = 15.5R \end{matrix}\right\}$.

c. By substitution $R = 8.6 - D$ So $\boxed{2.5D = 15.5(8.6 - D)}$ Therefore, $\boxed{D \approx 7.4 \text{ pounds.}}$

Problem 7



Heddy and Taylor are relaxing with you on their porch steps near a city sidewalk. They open identical glass cola bottles with twist-off metal lids. Then Heddy makes a proposal, "Let's toss these identical bottle caps onto the sidewalk. If they both land in matching positions, then I will win, and otherwise, you're the winner. What do you say?"

Taylor asks, "What do you mean by "matching positions'"?

Heddy answers, "The caps have an inside and an outside. If both the insides are up or both insides are down, those are matching."

"I don't know about that bet. There are three things that can happen: both insides are up, both are down or one of each. You can win two ways, but I can win only one way. Your chances are twice as good as mine," says Taylor.

"No, no, no, your so-called 'one way' happens twice as often as either of my ways, so our odds are equal," says Heddy.

Taylor asks you, "What do you think? And why do you think that way about these bottle caps?"

a. Is Taylor correct about Heddy's chances being twice as good as Taylor's? Justify your answer.

b. Is Heddy correct that 'matching' and 'not matching' are equally likely? Justify your answer.

c. Ignoring any morality issues on wagering, would you advise Taylor to accept the bet? Why or why not? Justify your answer.

Answers:

a. There are four possible outcomes, both insides up, both insides down, and two possibilities for one of each, depending on which cap is the up one. These four outcomes may not be equally likely, so we really cannot be sure about the "twice as likely" claim. More in part b.

b. Are "matching" and "not matching" equally likely? Just maybe. Let p be the probability that a single cap lands inside up. Then $1-p$ is the probability that it lands inside down. Both p and $1-p$ are between 0 and 1. Since the cap results are independent, the probability that both caps land inside up is p^2, and the probability that both land inside down is $(1-p)^2$. Since "both up" and "both down" are mutually exclusive events, the probability of the caps matching is $p^2 + (1-p)^2$. Plotting this probability function for values of p between 0 and 1 shows that the function is always at least 0.5. That minimum happens only if $p = 1/2$. So "matching" and "not matching' are equally likely only when $p = 1/2$ – like "fair" coins. For any other value of p Heddy would be more likely to win than Taylor.

c. Hey, Taylor, don't take the bet; you will probably lose more often than you win!

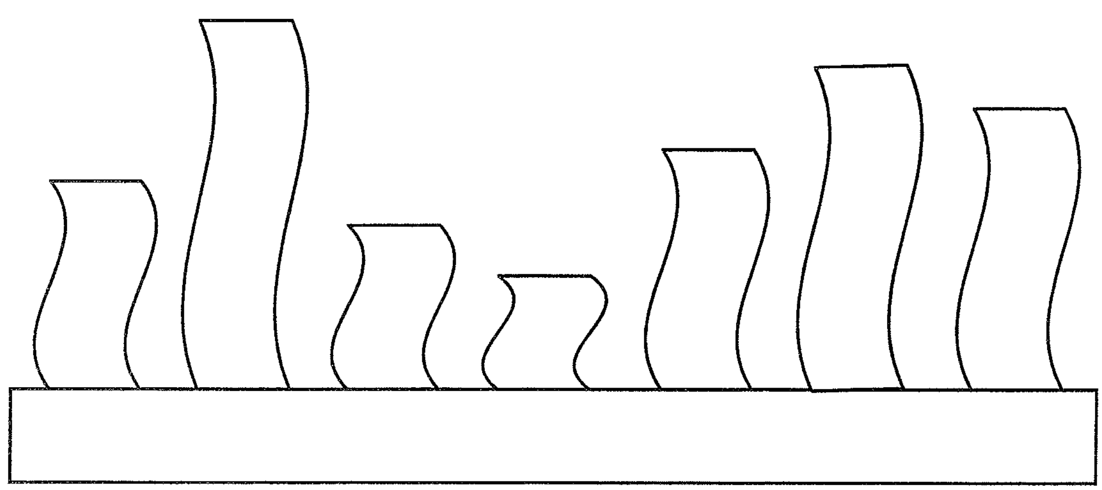
Problem 8

In "Harry Potter and the Sorcerer's Stone", by JK Rowling, the three best friends Harry, Hermione, and Ron are working to protect the sorcerer's stone from Lord Voldemort. To get to the stone, they must pass through several obstacles, each one put in place by a different Hogwarts teachers.

Unfortunately, Ron is taken out by McGonagall's chess pieces. Thus we find our two heroes Harry and Hermione facing Professor Snape's challenge alone. They pass from the chessboard room into another room, where immediately flames appear in both the doorway behind them and the one in front of them. They see a table with seven different size bottles (as shown below) and are given the following poem[1].

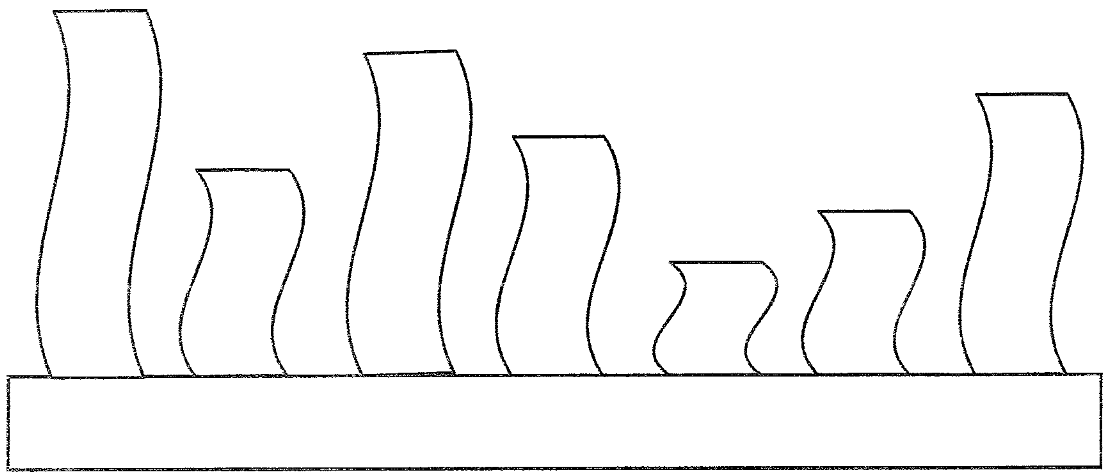
> Danger lies before you, while safety lies behind,
> Two of us will help you, whichever you would find,
> One among us seven will let you move ahead,
> Another will transport the drinker back instead,
> Two among our number hold only nettle wine,
> Three of us are killers, waiting hidden in line.
> Choose, unless you wish to stay here forevermore,
> To help you in your choice, we give you these clues four:
> First, however slyly the poison tries to hide
> You will always find some on nettle wine's left side;
> Second, different are those who stand at either end,
> But if you would move onward, neither is your friend;
> Third, as you see clearly, all are different size,
> Neither dwarf nor giant holds death in their insides;
> Fourth, the second left and the second on the right
> Are twins once you taste them, though different at first sight.

(a) Identify each of the bottles as containing one of Poison (P), Nettle Wine (N), a potion to go Back through the flame behind them (B), or a potion to go Forward through the flame in front of them (F).



[1] Rowling, J.K.. *Harry Potter and the Sorcerer's Stone*. New York: Scholastic, 1997. Print.

(b) For the second arrangement of bottles shown below, it is not possible to uniquely identify each of the bottles as containing one of Poison (P), Nettle Wine (N), a potion to go Back through the flame behind them (B), or a potion to go Forward through the flame in front of them (F). Find all possible orderings of the potions for this arrangement of bottles.



(c) Avid Harry Potter fans will remember that no picture displaying the relative sizes of the bottles is given in the book. Assume now that you do not know the size of any of the bottles arranged on the table. Find all of the possible orders of the potions that meet the requirements (except the one for size).

Answers.

If N is allowed to be the very left potion then the solutions are:

(a)

PNPFPNB

(b)

BPNPFPN	BPPNFPN	BPFPNPN
NPNFBPP	NPNBFPP	NPNPFPB
NPPNFPB	NPFPNPB	

(c)

BPNPFPN	BPPNFPN	BPFPNPN
PPNFBPN	PPNBFPN	
BPNFPPN		
PNPFPNB	PNFPPNB	
NPFPNPB	NPNFBPP	NPNBFPP
NPNPFPB	NPNFPPB	NPPNFPB

For the grading, if a student never had N on the left, then the problem was graded using only the 1, 3, and 8 solutions without N on the left. If any of the student's potion distributions had N on the left, then the problem was graded using the complete list.