2016

MARSHALL UNIVERSITY MATHEMATICS COMPETITION

Solutions

1. It is $\pi^2 / 8$. From $1 + 1/4 + 1/9 + 1/16 + \ldots + 1/n^2 + \ldots = \pi^2 / 6$, we have that
$\frac{1}{4}(1 + 1/4 + 1/9 + 1/16 + \ldots + 1/n^2 + \ldots) = \pi^2 / 24$ or that
$1/4 + 1/16 + 1/36 \ldots + 1/(2n)^2 + \ldots = \pi^2 / 24$. Subtracting this from the first equation gives
$1 + 1/9 + 1/25 + 1/49 + \ldots + 1/(2n+1)^2 + \ldots = \pi^2 / 6 - \pi^2 / 24 = \pi^2 / 8$.

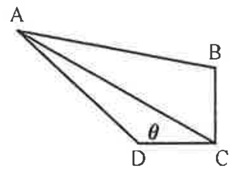
2. a. The region between the vertical lines $x = -\frac{1}{2}$ and $x = 0$ is the non-intersecting union of a 30-60-90 right triangle and a circular sector with central angle 30°. In radians these angles are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \& \frac{\pi}{6}$, respectively. The triangle has area = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$. The area of a circular sector is $\frac{1}{2} \cdot r^2 \cdot \theta$, where the angle is measured in radians. So the above circular sector has area = $\frac{1}{2} \cdot 1^2 \cdot \frac{\pi}{6} = \frac{\pi}{12}$. Total area to the left of the y-axis is thus $\boxed{\frac{\sqrt{3}}{8} + \frac{\pi}{12}}$.

Similarly, the region to the right of the y-axis is the non-intersecting union of a 30-60-90 right triangle and a circular sector with central angle 60°. The triangular area plus the sector area sums to $\boxed{\frac{\sqrt{3}}{8} + \frac{\pi}{6}}$.

The two boxed areas sum to $\boxed{\frac{\sqrt{3} + \pi}{4}}$. It could also be noted that the two hypotenuses form a right-angle sector. Then add the two triangular areas.

b. The area below the curve between $x = \frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{3}}{2}$ can be found by finding the area between the y-axis and each x-value, then subtracting. From part a, the area between the y-axis and $x = \frac{\sqrt{3}}{2}$ is $\frac{\sqrt{3}}{8} + \frac{\pi}{6}$. Between the y-axis and $x = \frac{\sqrt{2}}{2}$ is an equilateral right triangle plus a circular sector with central angle 45°. The sector is one eighth of a circle, so it has area $\frac{\pi}{8}$. And the triangle has area ¼. So the final result is $\frac{\sqrt{3}}{8} + \frac{\pi}{6} - \left(\frac{1}{4} + \frac{\pi}{8}\right) = \boxed{\frac{\pi}{24} + \frac{\sqrt{3} - 2}{8}}$.

3.



Strategy for solution: Let v be in units of miles/minute.
We know

$$30 = \frac{AC}{v}$$

$$35 = \frac{AB}{v} + BC = \frac{AB}{v} + 10$$

$$40 = \frac{AD}{v} + DC = \frac{AD}{v} + 10$$

So $AC = 30v,\ AB = 35v - 10,\ AD = 40v - 10$ gives three equations with 4 unknowns $AC,\ AB,\ AD,\ and\ v$.

From (iv) we know BCD is an isosceles right triangle with hypotenuse $DB = \sqrt{2} \cdot 10$.
We introduce a fifth unknown, angle $\theta = \angle ACD$.
We use the Law of Cosines to get a relationship between side AD and known sides $DC\ and\ AC$, and between AB and known sides $BC\ and\ AC$ as follows:

$$AD^2 = DC^2 + AC^2 - 2 \cdot DC \cdot AC \cdot \cos\theta = 100 + 900v^2 - 600v \cdot \cos\theta$$

$$AB^2 = BC^2 + AC^2 - 2 \cdot BC \cdot AC \cdot \cos\left(\frac{\pi}{2} - \theta\right) = 100 + 900v^2 - 600v \cdot \sin\theta$$

We can include these two equations in our earlier system of three equations to get a new system of five equations with θ as a 5th unknown. We can remove the first equation $AC = 30v$ to obtain four equations with 4 unknowns $AB,\ AD,\ v\ and\ \theta$. Solving the last two equations for $\cos\theta\ and\ \sin\theta$ and plugging into the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ results in the following equation: `600^2=(700-325*v)^2+(800-700*v)^2`

equivalently, $(7 - 3.25v)^2 + (8 - 7v)^2 = 36$

The solutions, after converting to mile/hour, are $v = 38.813\ and\ 119.814$.

4. 480 = cost of car rental

x = number of members on the trip

$x + 5$ = number of participants

$\left(\frac{480}{x} - 4.80\right)$ = fee per participant

Making $(x + 5)\left(\frac{480}{x} - 4.80\right) = 480$

$(x + 5)(480 - 4.8x) = 480x$

$480x - 4.8x^2 + 2400 - 24x = 480x$

$x^2 + 5x - 500 = 0$

$(x + 25)(x - 20) = 0$

a. 20 members joined the trip.

b. $19.20 fee per participant

c. $39.20 = fee + refreshment charge

5. a. (surface area of the 4 sides and the top) divided by 7

364 divided by 7

52 square inches

b. volume divided by 7

588 divided by 7

84 cubic inches

c. Diagrams will vary.

7. a. 27, 39, 78, 74, 18.5, 5

b. π, $\pi + 12$, $2\pi + 24$, $2\pi + 20$, $.5\pi + 5$, 5

c. $1 + 2i$, $13 + 2i$, $26 + 4i$, $22 + 4i$, $5.5 + i$, 5

d. The sequence cannot escape ending in 5. Let x be the starting number.

x, $x + 12$, $2x + 24$, $2x + 20$, $.5x + 5$, 5

(A) The probability that the first child will have the disorder can be found with the help of the following tree diagram.

50%	Agressive	60%	Disorder	▶ 60% of 50% = 30%
		40%	No Disorder	▶ 40% of 50% = 20%
50%	Passive	20%	Disorder	▶ 20% of 50% = 10%
		80%	No Disorder	▶ 80% of 50% = 40%

The probability of having the disorder is 30% + 10% = 40%.

(B) Once it is known that the first child has the disorder, this will provide enough information to update the probabilities for the two forms. This can be seen from the table below.

	First child has the disorder	First child does not have the disorder	Total
Agressive	30%	20%	50%
Passive	10%	40%	50%
Total	40%	60%	100%

If we know that the first child has the disorder, then we are restricted to the first column of the table. So we see that …

… the probability that John's disorder is aggressive is $\frac{30\%}{40\%} = 75\%$

… the probability that John's disorder is passive is $\frac{10\%}{40\%} = 25\%$

So if Jane and John decide to have another child, the probability that this child will have the disorder can be obtained with the help of the following tree diagram.

75%	Agressive	60%	Disorder	▶ 60% of 75% = 45%
		40%	No Disorder	▶ 40% of 75% = 30%
25%	Passive	20%	Disorder	▶ 20% of 25% = 5%
		80%	No Disorder	▶ 80% of 25% = 20%

The probability that the second child will have the disorder is 45% + 5% = 50%.

8

(A) ${}_8C_4 = \dfrac{8!}{4!4!} = 70$

(B) Each of the arrangements in (A) would correspond to 2 of these arrangements. For example, if the children are split into 1–4 and 5–8, that would correspond to having children 1–4 finger painting and the rest are modeling clay, or to having children 1–4 modeling clay and the rest finger painting.

So the number of such groupings would be $\dfrac{70}{2} = 35$.

(C) The group of 4 is unique due to its size. So …

Step1: Select the group of 4. → ${}_8C_4 = \dfrac{8!}{4!4!} = 70$

Step 2: Split the remaining 4 children into groups of 2 each. → $\dfrac{{}_4C_2}{2} = \dfrac{6}{2} = 3$

Step 3: Multiply. → 70 (3) = 210

(D) If the groups would be engaging in different activities, then the number of arrangements would be

$$\frac{30!}{10!10!10!}$$

But if all three groups are engaging in the same activity, then each of the above arrangements corresponds to 3! of these arrangements.

So the number is given by $\dfrac{30!}{3!10!10!10!}$